AMERICAN UNIVERSITY OF BEIRUT
Math 218
Elementary Linear Algebra – Sections 4 & 9
Fall 2009-2010

QUIZ 1, November 10

Solution

Duration: 70 minutes

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Student name	
Section	
AUB ID #	

Exercise	Out of	GRADE
1	15	
2	10	
3	15	
4	15	
5	20	
6	25	
TOTAL	100	

INSTRUCTIONS

1. Simple scientific calculators are allowed. Programmable calculators are NOT allowed.
2. Write your answers on the question sheet.
3. Partial solution or partial justification will receive partial credit.
4. The back of each page can be used if needed, and there is one extra blank sheet at the end. Whenever extra space is used, always indicate clearly where the grader should continue reading.

Exercise 1: (15 points)

Consider the linear system: $\begin{cases} ax + \qquad bz = 2 \\ ax + ay + 4z = 4 \\ \qquad ay + 2z = b \end{cases}$

Find for what values of a and b the system has:

a) a unique solution

b) a one-parameter family of solutions

c) a two-parameter family of solutions

d) no solution

Write the augmented matrix associated to the linear system and reduce it to a row echelon form:

$$\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & a & 2 & b \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & b-2 & b-2 \end{array}\right)$$

a) Case 1: $\boxed{a \neq 0}$ and $\boxed{b \neq 2}$

Then the REF has no rows of 0's and the 3x3 linear system has a unique solution

b) Case 2: $\boxed{a \neq 0}$ and $\boxed{b = 2}$

The REF is then: $\left(\begin{array}{ccc|c} a & 0 & 2 & 2 \\ 0 & a & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$

The linear system has a one-parameter family of solutions

c) Case 3 $\boxed{a = 0}$ and $\boxed{b = 2}$

The REF becomes: $\left(\begin{array}{ccc|c} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\longrightarrow \left(\begin{array}{ccc|c} 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ The linear system has a two-parameter family of solutions

d) Case 4 $\boxed{a = 0}$ and $\boxed{b \neq 2}$

The linear system is then inconsistent, since the REF will have 2 rows of 0's augmented with a nonzero entry:

$$\left(\begin{array}{ccc|c} 0 & 0 & b & 2 \\ 0 & 0 & 4-b & 2 \\ 0 & 0 & b-2 & b-2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 0 & 0 & b & 2 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & -2 & b-4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 0 & 0 & b & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & b-4 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 0 & b & 2 \\ 0 & 0 & -2 & b-4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2-b \\ 0 & 0 & 0 & b-2 \end{array}\right)$$

Exercise 2: (10 points)

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

Check that A is invertible and write A as a product of elementary matrices.

$\det(A) = 4$, A is therefore invertible.
i.e. the reduced row echelon form of A is I.

$$\underbrace{\begin{pmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \xrightarrow{\frac{1}{4}R_2 \to R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3/4 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 - \frac{3}{4}R_3 \to R_2]{R_1 + 2R_3 \to R_1} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I}$$

Performing every elementary row operation on A is equivalent to multiplying A on the left by the elementary matrix obtained by performing on I the same row operation: $E_3 E_2 E_1 A = I$

where E_1 is obtained by performing on I: $\frac{1}{4}R_2 \to R_2$
E_2 " " " " " : $R_1 + 2R_3 \to R_1$
E_3 " " " " " : $R_2 - \frac{3}{4}R_3 \to R_2$

It follows: $A = E_1^{-1} E_2^{-1} E_3^{-1}$

where the inverses E_1^{-1}, E_2^{-1} and E_3^{-1} are elementary matrices obtained by performing on I the inverse row operations,
i.e: E_1^{-1} : $4R_2 \to R_2$,
E_2^{-1} : $R_1 - 2R_3 \to R_1$
E_3^{-1} : $R_2 + \frac{3}{4}R_3 \to R_2$

Therefore:

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1^{-1}} \underbrace{\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_2^{-1}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3/4 \\ 0 & 0 & 1 \end{pmatrix}}_{E_3^{-1}}$$

$A = E_1^{-1} E_2^{-1} E_3^{-1} = E_1^{-1} E_3^{-1} E_2^{-1} = E_2^{-1} E_1^{-1} E_3^{-1}$

Exercise 3: (15 points)

Let A be the 3x3 matrix: $A = \begin{pmatrix} 1 & 3 & 4 \\ 3 & -1 & 6 \\ -1 & 5 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Show that A is invertible and find A^{-1}.

Deduce the solution of the linear system $AX = b$.

* Get first the reduced row echelon form of A, and perform the same elementary row operations on I:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 3 & -1 & 6 & 0 & 1 & 0 \\ -1 & 5 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -10 & -6 & -3 & 1 & 0 \\ 0 & 8 & 5 & 1 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3/5 & 3/10 & -1/10 & 0 \\ 0 & 8 & 5 & 1 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 11/5 & 1/10 & 3/10 & 0 \\ 0 & 1 & 3/5 & 3/10 & -1/10 & 0 \\ 0 & 0 & 1/5 & -7/5 & 4/5 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 11/5 & 1/10 & 3/10 & 0 \\ 0 & 1 & 3/5 & 3/10 & -1/10 & 0 \\ 0 & 0 & 1 & -7 & 4 & 5 \end{array}\right) \longrightarrow \left(\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I} \middle| \underbrace{\begin{array}{ccc} 31/2 & -17/2 & -11 \\ 9/2 & -5/2 & -3 \\ -7 & 4 & 5 \end{array}}_{A^{-1}}\right)$$

Since the RREF of A is I, then A is invertible

and $A^{-1} = \begin{pmatrix} 31/2 & -17/2 & -11 \\ 9/2 & -5/2 & -3 \\ -7 & 4 & 5 \end{pmatrix}$

* $AX = b \iff X = A^{-1}b = \begin{pmatrix} 31/2 & -17/2 & -11 \\ 9/2 & -5/2 & -3 \\ -7 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\therefore \quad X = \begin{pmatrix} 9/2 \\ 3/2 \\ -2 \end{pmatrix}$

Exercise 4: (15 points)

Evaluate the determinant of the following matrix:

$$A = \begin{pmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{pmatrix}$$

Let us combine row reduction and cofactor expansion:

$$|A| = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 12 & 0 & -1 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} 1 & -9 & -2 \\ 0 & -3 & -1 \\ 12 & 0 & -1 \end{vmatrix} = 1(3-0) - 0 + 12(9-6)$$

yielding: $\underline{\det(A) = 39}$

Exercise 5: (20 points)

Let $L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 2 & 4 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$.

a) Find the matrix $A = LU$ and write the linear system $Ax = b$.

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 2 & 4 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 2 & -1 & 3 \\ -4 & 5 & 0 \\ 4 & 2 & 18 \end{pmatrix}}_{A \,=\, LU}$$

$$Ax = b \iff \begin{cases} 2x_1 - x_2 + 3x_3 = 1 \\ -4x_1 + 5x_2 = -2 \\ 4x_1 + 2x_2 + 18x_3 = 0 \end{cases}$$

b) Solve the linear system $Ax = b$ by using this LU-factorization of A.

$Ax = b \iff LUx = b$

let $Ux = y$

then $Ly = b$

* Step 1: Solve for y the triangular system $Ly = b$

$$\begin{cases} y_1 = 1 \\ -2y_1 + 3y_2 = -2 \\ 2y_1 + 4y_2 + y_3 = 0 \end{cases}$$

using forward substitution yields $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

* Step2: Solve for x the triangular linear system $Ux = y$

$$\begin{cases} 2x_1 - x_2 + 3x_3 = 1 \\ x_2 + 2x_3 = 0 \\ 4x_3 = -2 \end{cases}$$

Using backward substitution:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7/4 \\ 1 \\ -1/2 \end{pmatrix}$$

Exercise 6: (25 points) (5 points for each question)

a) Show that if $A^t A = A$, then A is symmetric and $A = A^2$.

* Assume $A^t A = A$
then $A^t = (A^t A)^t = A^t (A^t)^t = A^t A = A$
i.e.: A is <u>symmetric</u>.

* $A = A^t A$
and $A = A^t$ (symmetric) } therefore $A = AA \Leftrightarrow \underline{A = A^2}$

b) Prove or disprove: If $A^T A = I$, then $det(A) = 1$.

If $A^t A = I$ then $\det(A^t A) = \det(I)$

$$\det(A^t)\det(A) = 1$$
$$\det(A)\det(A) = 1$$
$$[\det(A)]^2 = 1$$
$$\Leftrightarrow \underline{\det(A) = 1} \quad \text{or} \quad \underline{\det(A) = -1}$$

The statement is disproved.

c) Any square matrix A satisfying $A^n = 0$ for some positive integer n, is called *nilpotent of order n*. Prove that such matrix could not be invertible.

$A^n = 0 \implies \det(A^n) = 0$ (since the determ. of the zero-matrix is 0)

But $\det(A^n) = \det(AA \cdots A)$
$= \det(A)\det(A) \cdots \det(A)$
$= [\det(A)]^n$

It follows $[\det(A)]^n = 0$ yielding $\det(A) = 0$
$\therefore$ A could not be invertible.

d) Show that if a square matrix A is such that $A^2 - 3A + I = 0$, then $A^{-1} = 3I - A$.

$$A^2 - 3A + I = 0 \iff 3A - 3A^2 = I$$

It follows: $A\,(3I - A) = I$

i.e: A is invertible and $A^{-1} = 3I - A$

e) A is a square n x n matrix. B and C are invertible matrices of the same size, such that:

$$(BA)^T B^{-1}\left(CB^{-1}\right)^{-1} = I.$$

Prove that A is invertible and find A^{-1} in terms of B and C.

$$(BA)^t B^{-1} (CB^{-1})^{-1} = I$$

$$\iff A^t B^t \underbrace{B^{-1} B}_{I} C^{-1} = I$$

$$A^t B^t C^{-1} = I$$

$$(A^t B^t C^{-1})^t = I^t$$

$$\underbrace{(C^{-1})^t B}\, A = I$$

i.e. A is invertible and $A^{-1} = (C^{-1})^t B$